THE CALGARY MATHEMATICAL ASSOCIATION

29^{TH} JUNIOR HIGH SCHOOL MATHEMATICS CONTEST

April 27, 2005

NAME: SOLUTIONS
PLEASE PRINT (First name Last name)

GENDER: ☐ M ☐ F

SCHOOL: ____________________ GRADE: ________ (7,8,9)

- You have 90 minutes for the examination. The test has two parts: PART A — short answer; and PART B — long answer. The exam has 9 pages including this one.
- Each correct answer to PART A will score 5 points. You must put the answer in the space provided. No part marks are given.
- Each problem in PART B carries 9 points. You should show all your work. Some credit for each problem is based on the clarity and completeness of your answer. You should make it clear why the answer is correct.

  PART A has a total possible score of 45 points.

  PART B has a total possible score of 54 points.
- You are permitted the use of rough paper. Geometry instruments are not necessary. References including mathematical tables and formula sheets are **not** permitted. Simple calculators without programming or graphic capabilities **are** allowed. Diagrams are not drawn to scale. They are intended as visual hints only.
- When the teacher tells you to start work you should read all the problems and select those you have the best chance to do first. You should answer as many problems as possible, but you may not have time to answer all the problems.

BE SURE TO MARK YOUR NAME AND SCHOOL AT THE TOP OF THIS PAGE.
THE EXAM HAS 9 PAGES INCLUDING THIS COVER PAGE.

Please return the entire exam to your supervising teacher at the end of 90 minutes.

MARKERS' USE ONLY							
PART A ________ $\times 5$	B1	B2	B3	B4	B5	B6	TOTAL (max: 99)

PART A: SHORT ANSWER QUESTIONS

A1 Boris asks you to lend him a certain amount of money between 1 cent and 10 cents inclusive. What is the smallest number of Canadian coins you need to have in order to be able to give Boris exactly what he asks you for, regardless of what it is?

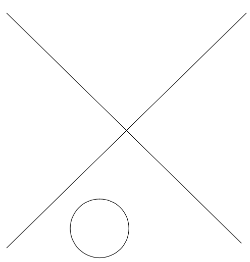
6

OR 1 dime, 1 nickel, 4 pennies
OR 1 nickel, 5 pennies

A2 Two prime numbers P and Q have the property that both their sum and their difference are again prime numbers. What are P and Q?

2 and 5

A3 In the figure, the two straight lines extend infinitely in both directions. How many circles could you draw that are tangent to the given circle and to both of the lines (that is, that just touch the circle and each of the lines)?



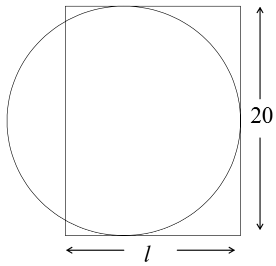
4

A4 Three cards each have one of the digits from 1 through 9 written on them. When the cards are arranged in some order they make a three-digit number. The largest number that can be made plus the second largest number that can be made is 1233.

What is the largest number that can be made?

621

A5 In the figure, the circle and the rectangle have the same area. What is the length l?



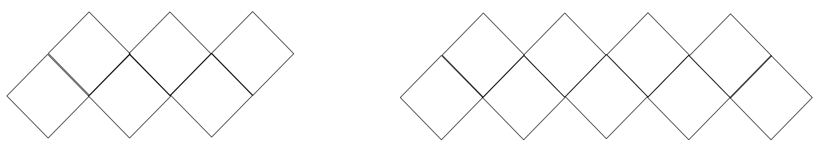
5π
≈ 15.7

A6. On each day that Adrian does his homework his mother gives him \$4, and on days he doesn't she takes \$1 away from him. After 30 days Adrian notices that he has the same amount of money as when he started even though he has spent nothing and had no other source of income. On how many of the 30 days did he do his homework?

6

A7 Below are two zig-zag shapes made of identical little squares 1 cm on a side. The first shape has 6 squares and a perimeter of 14 cm. The second has 9 squares and a perimeter of 20 cm. What is the perimeter of the zig-zag shape with 15 squares?

32

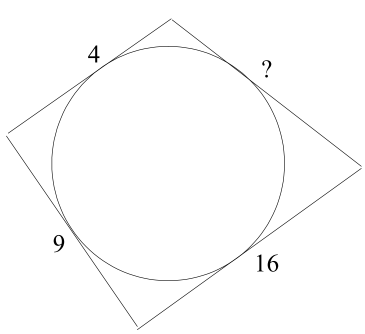


A8 You begin counting on your left hand starting with the thumb, then the index finger, the middle finger, the ring finger, then the little finger, and back to the thumb, and so on. (Thumb, index, middle, ring, little, ring, middle, index, thumb, index, ...) What is the 2005^{th} finger you count?

little

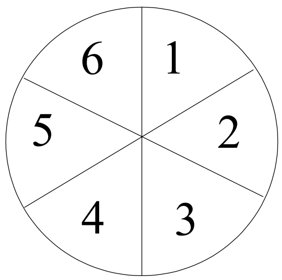
A9 A quadrilateral circumscribes a circle. Three of its sides have length 4, 9 and 16 cm, as shown. What is the length in cm of the fourth side?

11



PART B: LONG ANSWER QUESTIONS

B1 A pizza is cut into six pie-shaped pieces. Trung can choose any piece to eat first, but after that, each piece he chooses must have been next to a piece that has already been eaten (to make it easy to get the piece out of the pan). In how many different orders could he eat the six pieces?



SOLUTION:

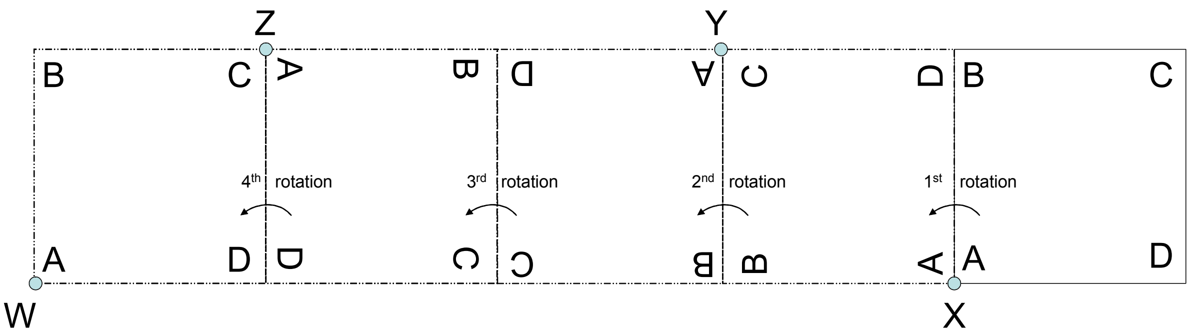
Trung can start with any one of the six pieces. After that, he will have only two choices for the next piece, so there are $6 \times 2 = 12$ ways for him to eat the first two pieces, namely:

piece 1 followed by piece 2	or piece 1 followed by piece 6
or piece 2 followed by piece 3	or piece 2 followed by piece 1
or piece 3 followed by piece 4	or piece 3 followed by piece 2
or piece 4 followed by piece 5	or piece 4 followed by piece 3
or piece 5 followed by piece 6	or piece 5 followed by piece 4
or piece 6 followed by piece 1	or piece 6 followed by piece 5.

For each of these choices, there will be two ways to eat the third piece, then two ways to eat the fourth piece, then two ways to eat the fifth piece, and finally only one choice for the last piece. We have to multiply by 2 each time we have two choices. So the total number of orders in which the pieces could be eaten is

$6 \times 2 \times 2 \times 2 \times 2 \times 1 =$ **96**.

B2 (a) A square of side length 1 metre, with corners labelled A, B, C, D as shown, is sitting flat on a table. It is rotated counterclockwise about its corner A through an angle of 90° (as shown in the figure), then rotated counterclockwise about B through 90°, then counterclockwise about C through 90°, and finally counterclockwise about D through 90°. After each rotation, how far away is the corner A from where it was at the beginning?

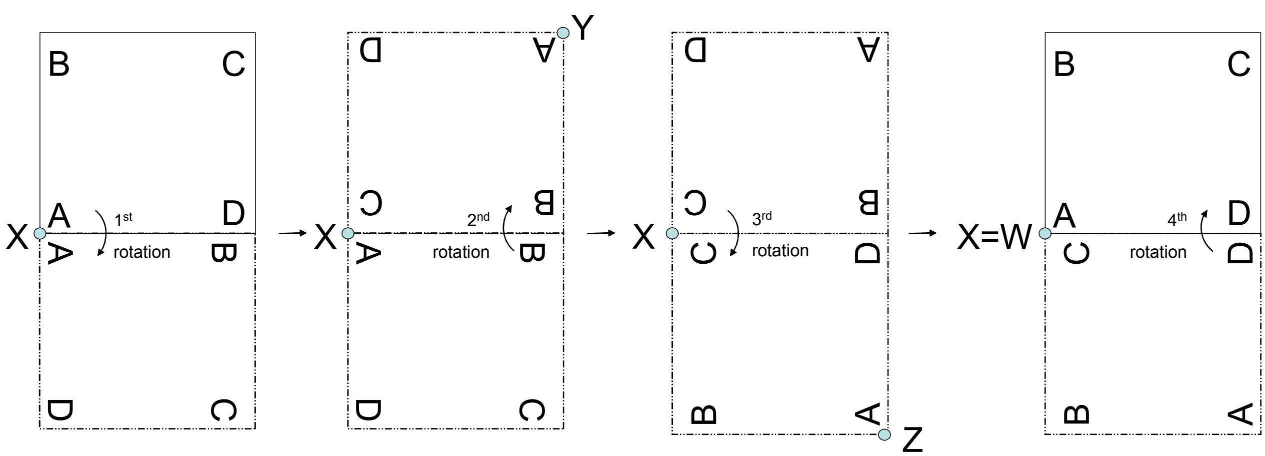


SOLUTION (a):

The position of the square after each rotation is shown in the diagram. Therefore the distance that corner A has moved is:

- after the first rotation: distance **0** metres;
- after the second rotation: distance $XY = \sqrt{1^2 + 1^2} = \sqrt{\mathbf{2}}$ (≈ 1.414) metres (by the Pythagorean Theorem);
- after the third rotation: distance $XZ = \sqrt{3^2 + 1^2} = \sqrt{\mathbf{10}}$ (≈ 3.162) metres;
- after the fourth rotation: distance $XW =$ **4** metres.

(b) The same question, only all the rotations are through 90° **clockwise**.



SOLUTION (b):

The position of the square after each rotation is shown in the diagrams. Therefore the distance that corner A has moved is:

- after the first rotation: distance **0** metres;
- after the second rotation: distance $XY = \sqrt{1^2 + 1^2} = \sqrt{\mathbf{2}}$ (≈ 1.414) metres;
- after the third rotation: distance $XZ = \sqrt{\mathbf{2}}$ (≈ 1.414) metres;
- after the fourth rotation: distance $XW =$ **0** metres.

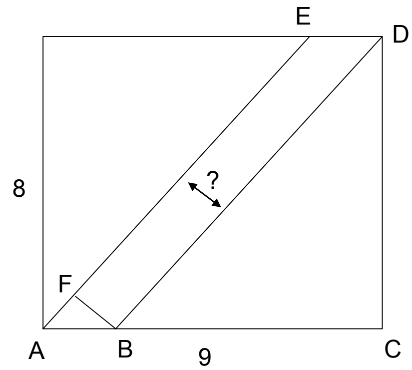
B3 Semra's score on a math test was recorded incorrectly by the teacher. Her real score was exactly four times the score that the teacher recorded. When the teacher corrected her mistake, the average score of the class went up by 2 points. There are 24 students in Semra's class (including Semra). What was Semra's real score on the math test ?

SOLUTION:

Since the average score of the class of 24 students went up by 2 points when the teacher corrected her mistake, the total of all the scores of the students must have gone up by $2 \times 24 = 48$ points. Thus 48 points must represent the difference between the originally recorded score and four times that score, that is, 48 must be equal to exactly three times the originally recorded score. So the originally recorded score must have been $48/3 = 16$, so Semra's real score must have been $16 \times 4 =$ **64**.

Of course, this question could also be done by algebra, or guess and check.

B4 The picture shows an 8 by 9 rectangle cut into three pieces by two parallel slanted lines. The three pieces all have the same area. How far apart are the slanted lines?



SOLUTION:

Since the area of the rectangle is $8 \times 9 = 72$, the area of each of the three pieces must be $72/3 = 24$. In particular the area of right triangle BCD must be 24. Since $CD = 8$ and the area of the triangle is $BC \times CD/2 = BC \times 4$, this means that $BC = 24/4 = 6$. By the Pythagorean Theorem, $BD = \sqrt{6^2 + 8^2} = 10$. Now the area of the parallelogram $ABDE$ is also 24, and equals the base times the altitude which is $BD \times BF = 10 \times BF$. Therefore BF, which is the distance between the slanted lines, is $BF = 24/10 = \mathbf{2.4}$.

Another way to do this problem is with similar triangles. Notice that the triangles AFB and BCD are both right triangles, and $\angle FAB = \angle DBC$ since AF and BD are parallel. Therefore triangles AFB and BCD are similar. Since we have (as above) that $BC = 6$ and $BD = 10$, and therefore $AB = AC - BC = 9 - 6 = 3$, we get by similar triangles that

$$\frac{FB}{CD} = \frac{AB}{BD} \qquad \text{which says} \qquad \frac{FB}{8} = \frac{3}{10},$$

and therefore $FB = 24/10 = 2.4$.

B5 (a) Find all the integers x so that

$$2 \le \frac{2005}{x} \le 5.$$

That is, find all integers x so that the fraction 2005 over x lies between 2 and 5 inclusive. How many such integers x are there?

SOLUTION (a): $2005/x$ will equal 5 when $x = 2005/5 = 401$, and $2005/x$ will equal 2 when $x = 2005/2 = 1002.5$. So the integers x that make $2005/x$ lie between 2 and 5 will be all the integers between 401 and 1002.5, namely

$$\mathbf{401, 402, 403, \ldots, 1002,}$$

and there are $1002 - 400 =$ **602** integers in this list.

(b) Find a positive integer N so that there are exactly 25 integers x satisfying

$$2 \le \frac{N}{x} \le 5.$$

SOLUTION (b): Depending on what N is, the integers x that work will be between the numbers $N/5$ and $N/2$, so the number of integers x that work will be approximately equal to

$$\frac{N}{2} - \frac{N}{5} = \frac{5N}{10} - \frac{2N}{10} = \frac{3N}{10}\ .$$

So we would like $3N/10$ to be about equal to 25: at least, this should be pretty close to a correct answer. Solving this equation, we get $3N = 250$ or $N = 250/3 \approx 83.3$. So try $N = 83$ to see if it works. We need $2 \le 83/x \le 5$, and the integers x which satisfy this inequality are the ones between $83/5 = 16.6$ and $83/2 = 41.5$, namely $x = 17, 18, 19, \ldots, 41$, and there are $41 - 16 = 25$ of them, so $N =$ **83** is one solution.

There are two other solutions, **80** and **82**. These work because:

For $N = 82$, we need $2 \le 82/x \le 5$, and the integers x which satisfy this inequality are the ones between $82/5 = 16.4$ and $82/2 = 41$, exactly the same 25 integers as for $N = 83$.

For $N = 81$, we need $2 \le 81/x \le 5$, and the integers x which satisfy this inequality are the ones between $81/5 = 16.2$ and $81/2 = 40.5$, so we lose 41 from the previous list and do not gain any integers, so there are only 24 integers x this time, not 25.

For $N = 80$, we need $2 \le 80/x \le 5$, and the integers x which satisfy this inequality are the ones between $80/5 = 16$ and $80/2 = 40$, so we get to add 16 to the previous list and do not lose any integers, so we are back up to 25 integers in total, which is what we want.

Note: By being a bit more careful with the algebra, we could find these other two solutions and show that there are no others. We know that the integers x that work will be between the numbers $N/5$ and $N/2$. The number of integers x between $N/5$ and $N/2$ is at least $\frac{N}{2} - \frac{N}{5} - 1 = \frac{3N}{10} - 1$ and at most $\frac{N}{2} - \frac{N}{5} + 1 = \frac{3N}{10} + 1$, so we need

$$\frac{3N}{10} - 1 \le 25 \qquad \text{and} \qquad 25 \le \frac{3N}{10} + 1.$$

Solving these two inequalities, we get $3N/10 \le 26$ which says $N \le 86\,\frac{2}{3}$, and $3N/10 \ge 24$ which says $N \ge 80$. So the only possible solutions are $N = 80$ to 86, and testing them all shows that only 80, 82 and 83 work.

B6 Amy, Bart and Carol are eating carrot sticks. Amy ate half the number that Bart ate, plus one-third the number that Carol ate, plus one. Bart ate half the number that Carol ate, plus one-third the number that Amy ate, plus two. Carol ate half the number that Amy ate, plus one-third the number that Bart ate, plus three. How many carrot sticks did they eat altogether?

SOLUTION:

If we let A, B, C stand for the number of carrot sticks eaten by Amy, Bart and Carol respectively, the problem says that:

$$A = \frac{B}{2} + \frac{C}{3} + 1,$$

$$B = \frac{C}{2} + \frac{A}{3} + 2,$$

and

$$C = \frac{A}{2} + \frac{B}{3} + 3.$$

Rather than try to solve these equations for A, B and C, if we just add them together we get that

$$A + B + C = \frac{A + B + C}{2} + \frac{A + B + C}{3} + (1 + 2 + 3),$$

which means that

$$\left(1 - \frac{1}{2} - \frac{1}{3}\right)(A + B + C) = 6,$$

or

$$\frac{1}{6}(A + B + C) = 6.$$

Therefore $A + B + C = 6 \times 6 = \mathbf{36}$, which is the answer we want.

By the way, if you do solve for A, B and C separately, which can be done with some effort, you get fractions of carrot sticks as answers:

$$A = \frac{822}{73}\,, \qquad B = \frac{882}{73}\,, \qquad C = \frac{924}{73}\,.$$