THE CALGARY MATHEMATICAL ASSOCIATION

30^{TH} JUNIOR HIGH SCHOOL MATHEMATICS CONTEST

April 26, 2006

NAME: SOLUTIONS

PLEASE PRINT (First name Last name)

GENDER: ☐ M ☐ F

SCHOOL: ____________________ GRADE: ________ (7,8,9)

- You have 90 minutes for the examination. The test has two parts: PART A — short answer; and PART B — long answer. The exam has 9 pages including this one.
- Each correct answer to PART A will score 5 points. You must put the answer in the space provided. No part marks are given.
- Each problem in PART B carries 9 points. You should show all your work. Some credit for each problem is based on the clarity and completeness of your answer. You should make it clear why the answer is correct.

  PART A has a total possible score of 45 points.

  PART B has a total possible score of 54 points.
- You are permitted the use of rough paper. Geometry instruments are not necessary. References including mathematical tables and formula sheets are **not** permitted. Simple calculators without programming or graphic capabilities **are** allowed. Diagrams are not drawn to scale. They are intended as visual hints only.
- When the teacher tells you to start work you should read all the problems and select those you have the best chance to do first. You should answer as many problems as possible, but you may not have time to answer all the problems.

BE SURE TO MARK YOUR NAME AND SCHOOL AT THE TOP OF THIS PAGE.
THE EXAM HAS 9 PAGES INCLUDING THIS COVER PAGE.

Please return the entire exam to your supervising teacher at the end of 90 minutes.

MARKERS' USE ONLY							
PART A ________ ×5	B1	B2	B3	B4	B5	B6	TOTAL (max: 99)

PART A: SHORT ANSWER QUESTIONS

A1 A prime number plus a perfect square equals 99. What is the prime number?

83

A2 The price of a TV (before tax) is a whole number of dollars and is the same in Alberta and in BC. That tax is 7% in Alberta and 15% in BC. After the tax is applied the TV costs \$10 more in BC than in Alberta. What is the before-tax price (in dollars) of a TV?

125

A3 A quadrilateral has three sides of lengths 5.5, 6.5 and 7.5 metres. The length of the fourth side in metres is a positive integer. How many possible lengths (in metres) can the fourth side have?

19

A4 Reim and Bindu are in a line with other students, waiting to see a movie. There are at most 30 students in the line.
Reim says: "There are three times as many students after me in this line than before me."
Bindu says: "There are four times as many students after me in this line than before me."
How many students are in the line?

21

A5 Anna, Bob and Carol ran a 1000m race and each ran at a (different) constant speed throughout the entire race. When Anna finished, Bob and Carol were at the 800m and 600m mark respectively. When Bob finished, how far (in metres) was Bob ahead of Carol?

250

A6. Thirty-one students who write a contest get all the integer grades from 70 through 100, with different students getting different grades. When one particular score is removed, the average of the 30 remaining scores is the same as the average of all 31 scores. What score has been removed?

85

A7 Robert was reading a book and was counting the number of 1s that appeared in the page numbers. He counted that there were 24 ones. If the book starts on page 1, how many pages does the book contain?

102

A8 All of the possible arrangements of the letters $MATH$ are used to form four letter codes. These codes are put in a list in alphabetical order. (So the first code in the list is $AHMT$.) What is the 7th code in this list?

HAMT

A9 The number $N = 111\ldots1$ consists of 2006 ones. It is exactly divisible by 11. How many zeroes are there in the quotient $\frac{N}{11}$?

1002

PART B: LONG ANSWER QUESTIONS

B1 Silvia needs to buy two shirts. Two stores, Shirt Check and Supershirts, sell the shirts she is searching for. Shirt Check's regular price is \$5 more than Supershirts'. However, Shirt Check has a special where if you buy one shirt at the regular price, you get the second shirt at 40% off the regular price. Supershirts is selling every shirt at 10% off the regular price. It turns out that the two shirts would cost Silvia exactly the same at Shirt Check as at Supershirts. What is the regular price of a shirt at Shirt Check?

SOLUTION:

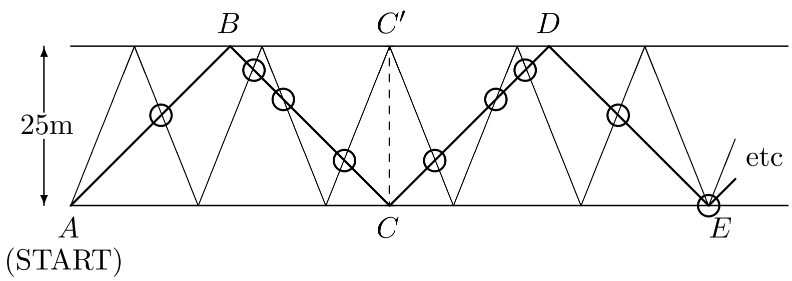
Let x be the regular price (in dollars) of a shirt at Shirt Check. Then $x - 5$ would be the regular price in dollars of a shirt at Supershirts. The cost of two shirts at Shirt Check would be $x + (x - .40x) = 1.6x$ dollars. The cost of two shirts at Supershirts would be $2[(x - 5) - .10(x - 5)] = 1.8(x - 5)$ dollars. Since the cost is the same at both stores, we get the equation $1.6x = 1.8(x - 5)$ which simplifies to $1.6x = 1.8x - 9$ or $0.2x = 9$, so $x = \mathbf{45}$ dollars is the cost of a shirt at Shirt Check.

You could also do this problem by guess and check.

B2 Alex swims $2\frac{1}{2}$ times as fast as Boris. They start together at one end of the pool and swim back and forth from one end to the other. The swimming pool is 25m long. Boris swims 30 lengths of the pool (750 m) and then stops. How many times has Alex passed Boris, either going in the same direction or in the opposite direction? (If Alex and Boris arrive at one end of the pool at the same time, it counts as a pass. But do not count the beginning when they start together.)

SOLUTION:

We draw a graph to show where the two boys are at all times. The horizontal direction is time and the vertical direction shows where they are in the pool at any time. The thick zigzag line is Boris and the thin zigzag line is Alex.



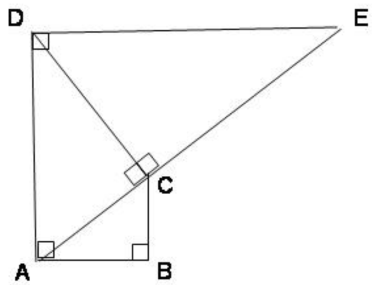
In the time that Boris swims two lengths of the pool (shown in the diagram as A to B to C), Alex swims 5 lengths of the pool and so is at the other end of the pool (at C') when Boris is at C. During this time they have passed each other four times, shown as the first four circled intersections between the thick zigzags and the thin zigzags. While Boris swims the next two lengths (C to D to E), they meet another four times (the next four circled intersections) and then meet at one end of the pool at E. This is the ninth time they pass each other, and Boris has swum four lengths.

Since Boris swims $30 = 4 \times 7 + 2$ lengths of the pool altogether, the above pattern will repeat a total of 7 times, and then the pattern A to B to C will happen one more time. Thus Boris and Alex pass each other a total of $9 \times 7 + 4 =$ **67** times.

B3 In the figure, $AB = 4$, $BC = 3$, and

$$\angle ABC = 90^\circ = \angle ACD = \angle DCE = \angle ADE = \angle DAB.$$

Find the length of AE.



SOLUTION:

By the Pythagorean Theorem, $AC = 5$.

Since $\angle ABC = 90^\circ$, we know that $\angle CAB + \angle ACB = 90^\circ$. But also $\angle CAB + \angle CAD = \angle BAD = 90^\circ$. Therefore $\angle ACB = \angle CAD$. Since $\angle ABC = \angle ACD$ $(= 90^\circ)$, this means that triangles ABC and DCA are similar. Thus

$$\frac{DA}{AC} = \frac{AC}{CB}\,, \qquad \text{or} \qquad \frac{DA}{5} = \frac{5}{3}\,,$$

and so $DA = 25/3$.

Since $\angle CAD = \angle DAE$ and $\angle ACD = \angle ADE$ $(= 90^\circ)$, this means that triangles ACD and ADE are similar. Thus

$$\frac{AE}{AD} = \frac{AD}{AC}\,, \qquad \text{or} \qquad \frac{AE}{25/3} = \frac{25/3}{5}\,,$$

and so $AE = (25/3)^2 \cdot (1/5) = \mathbf{125/9} = 13.\overline{8}$.

B4 Let's say that an integer $N > 1$ is *friendly* if every time N is written as the sum of *two* positive integers A and B, some digit of A or B is also a digit of N : that is, N **cannot** be written as the sum of two positive integers which do not use any of the digits of N. For example, 120 is **not** friendly, because you can write 120 as a sum of two positive integers without using the digits 1, 2 or 0 : for instance you could write 120 as $76 + 44$.

(a) Show that 2006 is not friendly.

SOLUTION:

(a) Any way of writing 2006 as the sum of two whole numbers not using the digits 2, 0 or 6 will work: for example, $2006 = 1999 + 7$ shows that 2006 is not friendly.

(b) Find an integer $N > 2006$ that is friendly. Make sure to say why you know your number is friendly.

SOLUTION:

(b) Any integer between 2007 and 2999 that contains the digit 1 is friendly (so for example, **2010** is friendly). The reason is that if you write such an integer as a sum of two positive integers, then one of these two integers must be at least 1000, so it starts with a 1 or a 2. For example, if $2010 = A + B$ where A and B are positive integers, then either A or B must be greater than 1000, so it must start with 1 or 2, both of which are digits of 2010.

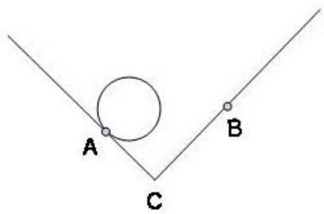
Another correct answer for part (b) would be any positive integer that contains all ten digits (for example the integer **1234567890** is friendly). The reason is simply that any time you write 1234567890 as a sum of two positive integers, you have to use some digits, and they are in 1234567890 whatever they are. As well, any correct answer for part (c) would also be a correct answer for part (b).

(c) Let's say that an integer $N > 1$ is *really friendly* if N cannot be written as the sum of two, or three, or four, or **any** number of positive integers without using at least one of the digits of N. Find a really friendly integer bigger than 1 but smaller than 100000. Make sure to say why you know your number is really friendly.

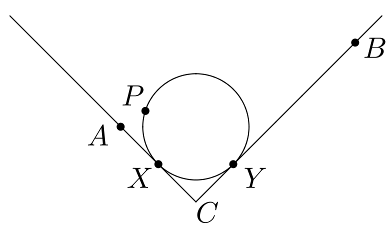
SOLUTION:

(c) Since the answer must be less than 100000, we can't just use the answer 1234567890 from part (b). But instead we could use any positive integer that uses all **odd** digits, for example **13579** is really friendly. The reason is that you can't possibly write the odd number 13579 as a sum of even integers, so whenever you write 13579 as a sum of positive integers, at least one of them must be odd and thus must end with one of the digits 1, 3, 5, 7 or 9.

B5 A wheel with radius 1 metre is rolled down one side of a right-angled trough (as in the diagram) and up the other side, without slipping. It runs over a blob of paint at point A on the first side. After that, every time that point on the wheel hits the trough it makes a paint mark on the trough. The first time this happens is at point B on the other side. The paint spot A on the first side is 2 metres from the corner C of the trough. How far from the corner is the paint mark B?



SOLUTION:



When the wheel hits the corner, it is touching the trough at two points X and Y as in the picture. Both XC and YC are equal to 1 metre, the radius of the wheel. Thus $AX = AC - XC = 2 - 1 = 1$ metre. Also, since X and Y are a quarter wheel apart, the length of the long (clockwise) arc from X to Y must be $\frac{3}{4} \cdot 2\pi = \frac{3\pi}{2}$ metres.

Let P be the spot on the wheel that hit the blob of paint. Since the wheel rolls without slipping, the length of the counterclockwise arc PX must equal the length AX which is 1 metre. Thus the length of the clockwise arc PY is $\frac{3\pi}{2} - 1$ metres. This must equal the length from Y to B. Therefore

$$BC = BY + YC = \left(\frac{3\pi}{2} - 1\right) + 1 = \boldsymbol{\frac{3\pi}{2}} \approx 4.712 \quad \text{metres.}$$

B6 Find all positive integers a and b so that

$$\frac{a}{b} - \frac{a+1}{b+1} = \frac{a+2}{b+2}.$$

Make sure to prove that you have found all solutions.

SOLUTION:

We can rewrite the equation as

$$\frac{a(b+1) - b(a+1)}{b(b+1)} = \frac{a+2}{b+2}$$

or

$$\frac{ab + a - ab - b}{b^2 + b} = \frac{a+2}{b+2}$$

or

$$\frac{a-b}{b^2+b} = \frac{a+2}{b+2} \ ,$$

and then cross multiply to get $(a-b)(b+2) = (a+2)(b^2+b)$. Multiplying this out gives $ab - b^2 + 2a - 2b = ab^2 + 2b^2 + ab + 2b$ which simplifies to

$$2a = ab^2 + 3b^2 + 4b. \qquad (1)$$

Both a and b are positive integers. Now if b were bigger than 1 then ab^2 would be bigger than $2a$, so $ab^2 + 3b^2 + 4b$ could not possibly be equal to $2a$. So for the equation (1) to be true, b must be equal to 1. Then equation (1) becomes $2a = a\cdot 1 + 3\cdot 1 + 4\cdot 1 = a + 7$, which has solution $a = 7$. Thus the only solution is $a = \mathbf{7}$, $b = \mathbf{1}$ which gives the correct equation

$$\frac{7}{1} - \frac{8}{2} = \frac{9}{3} \ .$$